THE WARING RANK OF THE SUM OF PAIRWISE COPRIME MONOMIALS

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Abstract. In this paper we compute the Waring rank of any polynomial of the form $F = \sum_{i=1}^{r} M_i$, where the M_i are pairwise coprime monomials, i.e., $GCD(M_i, M_j) = 1$ for $i \neq j$. In particular, we determine the Waring rank of any monomial. As an application we show that certain monomials in three variables give examples of forms of rank higher than the generic form. As a further application we produce a sum of power decomposition for any form which is the sum of pairwise coprime monomials.

1. Introduction

Let k be an algebraically closed field of characteristic zero and $k[x_1, \ldots, x_n]$ be the standard graded polynomial ring in n variables. Given a degree d form F the *Waring Problem for Polynomials* asks for the least value of s for which there exist linear forms $L_1, \ldots, L_s$ such that

$$F = \sum_{1}^{s} L_i^d \ .$$

This value of s is called the *Waring rank* of F (or simply the *rank* of F) and will be denoted by $\mathrm{rk}(F)$.

There was a long-standing conjecture describing the rank of a generic form F of degree d, but the verification of that conjecture was only found relatively recently in the famous work of J. Alexander and A. Hirschowitz [AH95]. However, for a given *specific* form F of degree d the value of $\mathrm{rk}(F)$ is not known in general. Moreover, in the general situation, there is no effective algorithmic way to compute the rank of a given form. Algorithms exist in special cases, e.g. when $n = 2$ for any d (i.e. the classical algorithm attributed to Sylvester) and for $d = 2$ any n (i.e. finding the canonical forms for quadratic forms).

Given this state of affairs, several attempts have been made to compute the rank of specific forms. One particular family of examples that has attracted attention is the collection of monomials.

A few cases where the ranks of specific monomials are computed can be found in [LM04] and in [LT10]. In [RS11] the authors determine $\mathrm{rk}(M)$ for the monomials

$$M = (x_1 \cdot \ldots \cdot x_n)^m$$

for any n and m. In particular, they show that $\mathrm{rk}(M) = (m+1)^{n-1}$. In this paper we completely solve the Waring Problem for monomials in Proposition 3.1 showing

that

$$(1)\quad \mathrm{rk}({x_1}^{a_1}\cdot\ldots\cdot {x_n}^{a_n}) = \frac{1}{a_1+1}\Pi_{i=1}^n(a_i+1) = \begin{cases} 1 & \text{for } n=1 \\ \Pi_{i=2}^n(a_i+1) & \text{for } n\geq 2 \end{cases},$$

where $1 \leq a_1 \leq \ldots \leq a_n$. A lengthier proof of this result was first obtained in [CCG11] and then, in a different form, in [BBT12].

Our approach to solving the Waring Problem for specific polynomials follows a well known path, namely the use of the Apolarity Lemma 2.1 to relate the computation of $\mathrm{rk}(F)$ to the study of ideals of reduced points contained in the ideal $F^\perp$. Using these ideas we obtained a complete solution to the Waring problem for polynomials which are the sum of coprime monomials, see Theorem 3.2. More precisely, if $F = M_1 + \ldots + M_r$ where the monomials M_i are such that $G.C.D.(M_i, M_j) = 1, i \neq j$ and $\deg(F) > 1$, then

$$\mathrm{rk}(F) = \mathrm{rk}(M_1) + \ldots + \mathrm{rk}(M_r).$$

Using our knowledge of the rank we obtained two interesting applications. We showed that, only in three variables and for degree high enough, certain monomials provide examples of forms having rank higher than the generic form, see Proposition 4.1. Finally, we find a minimal sum of powers decomposition for forms which are the sum of pairwise coprime monomials. In the case of monomials this result appeared in [CCG11] and was then improved in [BBT12].

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2. Basic facts

We consider k, an algebraically closed field of characteristic zero and the polynomial rings

$$S = k[x_{1,1},\ldots,x_{1,n_1},\ldots\ldots,x_{r,1},\ldots,x_{r,n_r}],$$
$$T = k[X_{1,1},\ldots,X_{1,n_1},\ldots\ldots,X_{r,1},\ldots,X_{r,n_r}].$$

We make T act via differentiation on S, e.g. we think of $X_{i,j} = \partial/\partial x_{i,j}$. (see, for example, [Ger96] or [IK99]). We refer to a polynomial in T as ∂, instead of using capital letters. In particular, for any form F in S_d we define the ideal $F^\perp \subseteq T$ as follows:

$$F^\perp = \{\partial \in T : \partial F = 0\}.$$

Given a homogeneous ideal $I \subseteq T$ we denote by

$$HF(T/I, i) = \dim_k T_i - \dim_k I_i$$

its *Hilbert function* in degree i. It is well known that for all $i >> 0$ the function $HF(T/I, i)$ is a polynomial function with rational coefficients, called the *Hilbert polynomial* of T/I. We say that an ideal $I \subseteq T$ is *one dimensional* if the Krull dimension of T/I is one, equivalently the Hilbert polynomial of T/I is some integer constant, say s. The integer s is then called the *multiplicity* of T/I. If, in addition, I is a radical ideal, then I is the ideal of a set of s distinct points. We will use the fact that if I is a one dimensional saturated ideal of multiplicity s, then $HF(T/I, i)$ is always $\leq s$.

Our main tool is the *Apolarity Lemma*, the proof of which can be found in [IK99, Lemma 1.31].

Lemma 2.1. *A homogeneous degree d form $F \in S$ can be written as*

$$F = \sum_{i=1}^{s} L_i^d, \quad L_i \text{ pairwise linearly independent}$$

if and only if there exists $I \subseteq F^\perp$ such that I is the ideal of a set of s distinct points.

We conclude with the following trivial, but useful, remark showing that the rank of a form does not vary by adding variables to the polynomial ring.

Remark 2.2. The computation of the rank of F is independent of the polynomial ring in which we consider F. To see this, consider a rank d form $F \in k[x_1, \ldots, x_n]$ and suppose we know $\mathrm{rk}(F)$. We can also consider $F \in k[x_1, \ldots, x_n, y]$ and we can look for a sum of powers decomposition of F in this extended ring. If

$$F(x_1, \ldots, x_n) = \sum_{1}^{r} \left(L_i(x_1, \ldots, x_n, y)\right)^d,$$

then, by setting $y = 0$, we readily get $r \geq \mathrm{rk}(F)$. Thus, by adding variables we can not get a sum of powers decomposition involving fewer summands. Moreover, if r is the minimal length of a sum of powers decomposition of F in the extended ring, we readily get $r = \mathrm{rk}(F)$. In particular, given a monomial

$$M = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n},$$

with $1 \leq a_1 \leq \ldots \leq a_n$ it is enough to work in $k[x_1, \ldots, x_n]$ in order to compute $\mathrm{rk}(M)$.

3. Main result

It is useful to recall the following. Let $I \subseteq T$ be an ideal and $\partial \in T_1$ a linear homogeneous differentiation. If ∂ is not a zero divisor in T/I then

$$HF(T/I, t) = \sum_{i=0}^{t} HF(T/(I + (\partial)), i)). \tag{2}$$

We first compute the rank of any monomial. Thus, we only consider the case $r = 1$ and, just for this result, we drop the double index notation, i.e. we abuse notation and we let $S = k[x_1, \ldots, x_n]$ and $T = k[X_1, \ldots, X_n]$.

Proposition 3.1. *Let $n \geq 1$ and $1 \leq a_1 \leq \ldots \leq a_n$. If*

$$M = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n},$$

then $\mathrm{rk}(M) = \frac{1}{a_1+1}\Pi_{i=1}^n (a_i + 1)$.

Proof. If $n = 1$, then M is the power of a variable and $\mathrm{rk}(M) = 1$; we can then assume $n > 1$. The perp ideal of M is $M^\perp = (X_1^{a_1+1}, \ldots, X_n^{a_n+1})$ and hence

$$I = (X_n^{a_n+1} - X_1^{a_n+1}, \ldots, X_2^{a_2+1} - X_1^{a_2+1}) \subseteq M^\perp.$$

As I is the ideal of a complete intersection scheme of $\frac{1}{a_1+1}\Pi_{i=1}^n (a_i + 1)$ distinct points, the Apolarity Lemma yields

$$\mathrm{rk}(M) \leq \frac{1}{a_1 + 1}\Pi_{i=1}^n (a_i + 1).$$

We now consider $I \subseteq M^\perp$ the ideal of a scheme of s distinct points; to complete the proof it is enough to show that $s \geq \frac{1}{a_1+1}\Pi_{i=1}^n(a_i+1)$. To do this, we set $I' = I : (X_1)$ and we notice that I' is the ideal of a scheme of $s' \leq s$ distinct points; notice that $s' > 0$ as $X_1 \notin I$. Clearly we have

$$I' + (X_1) \subseteq J = M^\perp : (X_1) + (X_1) = (X_1, X_2^{a_2+1}, \ldots, X_n^{a_n+1}).$$

Hence, for $t \gg 0$ we get

$$s' = HF(T/I', t) = \sum_{i=0}^{t} HF(T/(I' + (X_1)), i) \geq$$

$$\sum_{i=0}^{t} HF(T/J, i) = \frac{1}{a_1+1}\Pi_{i=1}^n(a_i+1)$$

where the last equality holds as J is a complete intersection ideal. The conclusion then follows as $s \geq s'$. $\square$

We now state and prove our main result.

Theorem 3.2. *Consider the degree d form*

$$F = M_1 + \ldots + M_r$$
$$= x_{1,1}^{a_{1,1}} \cdot \ldots \cdot x_{1,n_1}^{a_{1,n_1}} + \ldots\ldots + x_{r,1}^{a_{r,1}} \cdot \ldots \cdot x_{r,n_r}^{a_{r,n_r}},$$

where

$$a_{i,1} + \ldots + a_{i,n_i} = d, \quad 1 \leq a_{i,1} \leq \ldots \leq a_{i,n_i}, \quad (1 \leq i \leq r).$$

If $d = 1$ then $\mathrm{rk}(F) = 1$. If $d \geq 2$, then

$$\mathrm{rk}(F) = \sum_{i=1}^{r} \mathrm{rk}(M_i).$$

Proof. The case $d = 1$ is trivial as F is a linear form, thus we only have to prove the $d \geq 2$ case. For $d = 2$, F is a quadratic form. Since its associated matrix is congruent to a diagonal matrix of rank $\sum_{i=1}^r n_i$ the conclusion follows.

For $r = 1$ the form F is a monomial and the theorem is proved in Proposition 3.1.

Hence we have only to consider the cases $d > 2$ and $r > 1$.

By writing each monomial M_i as a sum of powers we get a sum of powers decomposition of F, thus we have $\mathrm{rk}(F) \leq \sum_{i=1}^r \mathrm{rk}(M_i)$. Hence, using Lemma 2.1, it is enough to show that if $F^\perp$ contains the ideal of a scheme of s distinct points, then $s \geq \sum_{i=1}^r \mathrm{rk}(M_i)$.

Let $I \subseteq F^\perp$ be the ideal of a scheme $\mathbb{X}$ of s distinct points, and let $\mathbb{X}' \subseteq \mathbb{X}$ be the subsets of the s' points of $\mathbb{X}$ not lying on $\{X_{1,1} = \cdots = X_{r,1} = 0\}$. Let I' be the ideal of $\mathbb{X}'$, i.e.,

$$I' = I : (X_{1,1}, \ldots, X_{r,1}).$$

We will prove that $s' \geq \sum_{i=1}^r \mathrm{rk}(M_i)$, so the conclusion will follow as $s \geq s'$.

The generic linear derivation $\alpha_1 X_{1,1} + \ldots + \alpha_r X_{r,1}$ (where $\alpha_i \in k$) is not a zero divisor in T/I'. Without loss of generality, and possibly rescaling the variables, we may assume that

$$\partial = X_{1,1} + \ldots + X_{r,1}$$

is not a zero divisor in T/I'. Hence, for $t \gg 0$ we get

$$s' = HF(T/I', t) = \sum_{i=0}^{t} HF(T/(I' + (\partial)), i). \tag{3}$$

Let w, $(0 \le w \le r)$, be the number of 1's in the set $\{a_{1,1}, \dots, a_{r,1}\}$. We may assume that

$$a_{1,1} = \cdots = a_{w,1} = 1.$$

We have

$$\begin{aligned} I' + (\partial) &\subseteq (F^\perp : (X_{1,1}, \dots, X_{r,1})) + (\partial) \\ &= (F^\perp : (X_{1,1})) \cap \ldots \cap (F^\perp : (X_{r,1})) + (\partial) \\ &\subseteq (x_{1,2}^{a_{1,2}} \cdot \ldots \cdot x_{1,n_1}^{a_{1,n_1}})^\perp \cap \ldots\ldots \cap (x_{w,2}^{a_{w,2}} \cdot \ldots \cdot x_{w,n_w}^{a_{w,n_w}})^\perp \\ &\cap((x_{w+1,1}^{a_{w+1,1}-1} \cdot x_{w+1,2}^{a_{w+1,2}} \cdot \ldots \cdot x_{w+1,n_{w+1}}^{a_{w+1,n_{w+1}}})^\perp + (X_{w+1,1})) \\ &\cap \ldots\ldots \cap ((x_{r,1}^{a_{r,1}-1} \cdot x_{r,2}^{a_{r,2}} \cdot \ldots \cdot x_{r,n_r}^{a_{r,n_r}})^\perp + (X_{r,1})) \\ &= J_1 \cap \ldots \cap J_r, \end{aligned}$$

where

$$J_1 = (X_{1,1}, X_{1,2}^{a_{1,2}+1}, \dots, X_{1,n_1}^{a_{1,n_1}+1}, X_{2,1}, \dots, X_{2,n_2}, \dots\dots, X_{r,1}, \dots, X_{r,n_r});$$

$$J_2 = (X_{1,1}, \dots, X_{1,n_1}, X_{2,1}, X_{2,2}^{a_{2,2}+1} \dots, X_{2,n_2}^{a_{2,n_2}+1}, \dots\dots, X_{r,1}, \dots, X_{r,n_r});$$

$$\vdots$$

$$J_r = (X_{1,1}, \dots, X_{1,n_1}, \dots\dots, X_{r,1}, X_{r,2}^{a_{r,2}+1} \dots, X_{r,n_r}^{a_{r,n_r}+1}).$$

Observe that, if $n_i = 1$, then J_i is the maximal ideal.

So we have

$$I' + (\partial) \subseteq J_1 \cap \ldots \cap J_r. \tag{4}$$

The only linear forms in $J_1 \cap \ldots \cap J_r$ are $X_{1,1}$, $X_{2,1}$, $\dots$, $X_{r,1}$, hence

$$\dim(J_1 \cap \ldots \cap J_r)_1 = r. \tag{5}$$

Now we will prove by contradiction that in I' there are no linear forms. Assume that $L \in I'$ is a linear form,

$$L = \alpha_{1,1}X_{1,1} + \ldots + \alpha_{1,n_1}X_{1,n_1} + \cdots + \alpha_{r,1}X_{r,1} + \cdots + \alpha_{r,n_r}X_{r,n_r}.$$

Since $I' = I : (X_{1,1}, \dots, X_{r,1})$ we have

$$X_{1,1}L, \dots, X_{r,1}L \in I.$$

Hence $X_{i,1}L \in F^\perp$, $(1 \le i \le r)$, so for $1 \le i \le w$ we get

$$\alpha_{i,2} = \ldots = \alpha_{i,n_i} = 0,$$

and for $i > w$ we get

$$\alpha_{i,1} = \ldots = \alpha_{i,n_i} = 0.$$

Hence

$$L = \alpha_{1,1}X_{1,1} + \ldots + \alpha_{w,1}X_{w,1}.$$

Let $\mathbb{X}'' = \mathbb{X} \setminus \mathbb{X}'$, that is, $\mathbb{X}''$ is the subsets of the points of $\mathbb{X}$ lying on $\{X_{1,1} = \cdots = X_{r,1} = 0\}$. Obviously $\{L = 0\} \supseteq \mathbb{X}''$. It follows that $L \in I$.

Since $I \subseteq F^\perp$, and in $F^\perp$ there are no linear forms, we get a contradiction.

So we have

$$\dim(I' + (\partial))_1 = 1. \tag{6}$$

Now by (3), (4), (5), (6), we get

$$\begin{aligned} s' &= \sum_{i=0}^{t} HF(T/(I' + (\partial)), i) \\ &= \dim T_1 - 1 + \sum_{i\neq 1;\ i=0}^{t} HF(T/(I' + (\partial)), i) \\ &\geq \dim T_1 - 1 + \sum_{i\neq 1;\ i=0}^{t} HF(T/J_1 \cap \ldots \cap J_r, i) \\ &= \dim T_1 - 1 + \sum_{i=0}^{t} HF(T/J_1 \cap \ldots \cap J_r, i) - (\dim T_1 - r). \end{aligned}$$

Hence

$$s' \geq \sum_{i=0}^{t} HF(T/J_1 \cap \ldots \cap J_r, i) + r - 1. \tag{7}$$

We now need the following claim.

Claim: For $t \gg 0$,

$$\sum_{i=0}^{t} HF(T/J_1 \cap \ldots \cap J_r, i) = \sum_{i=0}^{t} HF(T/J_1, i) + \ldots + \sum_{i=0}^{t} HF(T/J_r, i) - r + 1.$$

Proof of the Claim: To prove the claim we proceed by induction on r. If $r = 1$ the claim is obvious. Let $r > 1$ and consider the following short exact sequence:

$$0 \longrightarrow T/J_1 \cap \ldots \cap J_r \longrightarrow T/J_1 \oplus T/J_2 \cap \ldots \cap J_r \longrightarrow T/(J_1 + J_2 \cap \ldots \cap J_r) \longrightarrow 0.$$

By the inductive hypothesis, and since $J_1 + J_2 \cap \ldots \cap J_r$ is the maximal ideal, we get the conclusion. $\square$

Now we notice that for $t \gg 0$ and since the J_i are generated by regular sequences of length $n_1 + \cdots + n_r$, we have

$$\sum_{i=0}^{t} HF(T/J_1, i) = \frac{1}{a_{1,1} + 1} \Pi_{j=1}^{n_1} (a_{1,j} + 1) = \operatorname{rk}(M_1),$$

$$\vdots$$

$$\sum_{i=0}^{t} HF(T/J_r, i) = \frac{1}{a_{r,1} + 1} \Pi_{j=1}^{n_r} (a_{r,j} + 1) = \operatorname{rk}(M_r).$$

Hence by (7) and the claim the conclusion immediately follows.

$\square$

Remark 3.3. Let $F = \sum_1^r M_i$ be as in Theorem 3.2 and $\mathbb{X}$ be a set of s distinct points such that $I_{\mathbb{X}} \subset F^{\perp}$. If $\mathbb{X} \cap \{X_{1,1} = \cdots = X_{r,1} = 0\} = \mathbb{X}' \neq \emptyset$ is a set of s' points, by the proof of the theorem we see that $s \geq s' + 1 \geq \operatorname{rk}(F) + 1$. In particular, $\mathbb{X}$ does not have the least possible cardinality if it intersects the special linear space $\{X_{1,1} = \cdots = X_{r,1} = 0\}$.

4. Applications

We now present a few applications of our results.

4.1. On the rank of the generic form. It is well known, see [AH95], that for the generic degree d form in n variables F one has

$$\mathrm{rk}(F) = \left\lceil \frac{\binom{d+n-1}{d}}{n} \right\rceil .$$

However, the rank for a given specific form can be bigger or smaller than that number. Moreover, it is trivial to see that every form of degree d is a sum of $\binom{d+n}{d}$ powers of linear forms. But, in general, it is not known how big the rank of a degree d form can be.

Using monomials we can try to produce explicit examples of forms having rank bigger than that of the generic form. We give a complete description of the situation for the case of three variables. In that case, for $d \gg 0$, there are degree d monomials with rank bigger than that the generic form, see Proposition 4.1. However, for more than three variables, this is no longer the case, see Remark 4.2.

Proposition 4.1. *Let $n = 3$ and $d > 2$ be an integer. Then*

$$\max\{rk(M) : M \in S_d \text{ is a monomial}\} = \begin{cases} \left(\frac{d+1}{2}\right)^2 & d \text{ is odd} \\ \frac{d}{2}\left(\frac{d}{2}+1\right) & d \text{ is even} \end{cases}$$

and this number is asymptotically $\frac{3}{2}$ of the rank of the generic degree d form in three variables, i.e.

$$\max\{rk(M) : M \in S_d \text{ is a monomial}\} \simeq \frac{3}{2}\left\lceil \frac{\binom{d+2}{2}}{3} \right\rceil$$

for $d \gg 0$.

Proof. We consider monomials $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with the conditions $1 \le a_1 \le a_2 \le a_3$,

$$a_1 + a_2 + a_3 = d$$

and we want to maximize the function $f(a_2, a_3) = (a_2+1)(a_3+1)$. Considering a_1 as a parameter we are reduced to an optimization problem in the plane where the constraint is given by a segment and the target function is the branch of an hyperbola. For any given a_1, it is easy to see that the maximum is achieved when a_2 and a_3 are as close as possible to $\frac{d-a_1}{2}$. Also, when $a_1 = 1$ we get the maximal possible value. In conclusion $\mathrm{rk}(M)$ is maximal for the monomial

$$M = x_1 x_2^{\frac{d-1}{2}} x_3^{\frac{d-1}{2}} \,(d \text{ odd}) \text{ or } M = x_1 x_2^{\frac{d}{2}-1} x_3^{\frac{d}{2}} \,(d \text{ even}).$$

With a straightforward computation, one easily sees that the rank of the generic form is asymptotically $\frac{d^2}{6}$, while the maximal rank of a degree d monomial is asymptotically $\frac{d^2}{4}$. The conclusion follows.

□

Remark 4.2. If $n \ge 4$ and $d \gg 0$, the degree d monomials do not provide examples of high rank forms. For example, let $d = (n-1)k + 1$ and consider a highest rank degree d monomial

$$M = x_1 x_2^k \cdot \ldots \cdot x_n^k .$$

If F is a generic degree d form, then

$$\operatorname{rk}(M) \simeq \frac{n!}{(n-1)^{n-1}}\operatorname{rk}(F)$$

for $d \gg 0$ and we note that $\frac{n!}{(n-1)^{n-1}} \leq 1$ if $n \geq 4$. Hence, for each $n \geq 4$, there are infinitely many values of d for which no degree d monomial has rank bigger than the generic form.

4.2. **Sum of powers decomposition for polynomials.** Since we now know the rank of any given monomial, we can give a description of one of its minimal sum of powers decompositions. An explicit form for the scalars γ can be found in [BBT12] and it was also noticed by G. Whieldon [W11]. In Remark 4.5 we see how to use this to obtain a minimal sum of powers decomposition for the sum of coprime monomials.

Proposition 4.3. *For integers $1 \leq a_1 \leq \ldots \leq a_n$ consider the monomial*

$$M = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$$

and let $\mathcal{Z}(i) = \{z \in \mathbb{C} : z^{a_i+1} = 1\}$. Then

$$M = \sum_{\epsilon(i) \in \mathcal{Z}(i), i=2,\ldots,n} \gamma_{\epsilon(2),\ldots,\epsilon(n)} \left(x_1 + \epsilon(2)x_2 + \ldots + \epsilon(n)x_n\right)^d$$

where the $\gamma_{\epsilon(2),\ldots,\epsilon(n)}$ are scalars and this decomposition involves the least number of summands.

Proof. Another consequence of [IK99, Lemma 1.15] allows one to write a form as a sum of powers of linear forms. If $I \subset M^{\perp}$ is an ideal of s points, then

$$M = \sum_{j=1}^{s} \gamma_j \left(\alpha_j(1)x_1 + \alpha_j(2)x_2 + \ldots + \alpha_j(n)x_n\right)^d$$

where the γ_j are scalars and $[\alpha_1 : \ldots : \alpha_n]$ are the coordinates of the points having defining ideal I. Given M we can choose the following ideal of points

$$I = (y_2^{a_2+1} - y_1^{a_2+1}, y_3^{a_3+1} - y_1^{a_3+1}, \ldots, y_n^{a_n+1} - y_1^{a_n+1}).$$

It is straightforward to see that the points defined by I have coordinates

$$[1 : \epsilon(2) : \ldots : \epsilon(n)]$$

where $\epsilon(i) \in \mathcal{Z}(i)$. Renaming the scalars and taking all possible combinations of the roots of 1 we get the desired $\operatorname{rk}(M) = \Pi_{i=2}^{n}(a_i+1)$ points and the result follows. □

Remark 4.4. In order to find an explicit decomposition for a given monomial it is enough to solve a linear system of equations to determine the γ_j. For example, in the very simple case of $M = x_0x_1x_2$, we only deal with square roots of 1 and we get:

$$x_0x_1x_2 = \frac{1}{24}(x_0+x_1+x_2)^3 - \frac{1}{24}(x_0+x_1-x_2)^3 - \frac{1}{24}(x_0-x_1+x_2)^3 + \frac{1}{24}(x_0-x_1-x_2)^3.$$

Remark 4.5. Using Proposition 4.3 we can easily find a minimal sum of powers decomposition for the sum of coprime monomials. If $F = M_1 + \ldots + M_r$, then a minimal sum of powers decomposition of F is obtained by decomposing each M_i as described in Proposition 4.3.

Remark 4.6. Let $F = \sum_1^r M_i$ be the sum of coprime monomials, and $F = \sum_1^{\mathrm{rk}(F)} L_i^d$ be a minimal sum of powers decomposition of F. By Remark 3.3 we get that each linear form L_i must involve the variable $X_{1,i}, i = 1, \ldots, r$, where these are the variables with the least exponent in each M_i. A particular instance of this property, for $r = 1$, has been noticed in [BBT12]

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